Impulsive stabilization of two kinds of third-order delay differential equations

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Abstract: These instructions provide you guidelines for preparing papers for International Journal. Use this document as a template and as an instruction set. This paper is concerned with the impulsive stabilization problems for two kinds of 3th-order delay differential equations. By the method of Lyapunov function, we prove that the non-impulsive equations can be stabilized by the proper impulse control. Our results has improved and extended some results. We also give examples to illustrate the efficiency of our results.

Keywords:Third-order differential equation; Impulsive stabilization; Delay; Lyapunov function

1. Introduction

When you submit your paper print it in two-column format, including figures and tables. In addition, designate one author as the "corresponding author". This is the author to whom proofs of the paper will be sent. Proofs are sent to the corresponding author only. Third-order differential equation; Impulsive stabilization; Delay; Lyapunov function Recently, the problem of impulsive stabilization for differential equations has attracted many authors' attentions and some results have been published (see [1-10]). Impulses can make unstable systems stable. The problem of stabilizing the solutions by imposing proper impulse control has been used in many fields such as physics, pharmacokinetics, biotechnology, economics, chemical technology. However some authors have researched the impulsive stabilization problems for two kinds of 2th-order delay differential equations in [1-5], they proved that it also can be made exponentially continuous with respect to initial data by impulses on some interval t_k. And the presented references here ([7,8,10]) dealt with mostly the first-order delay differential equations (see[7]), In this paper , we consider third-order delay differential equations and deal with more general equations, the results we prove here generalize recent ones by Li and Weng [1]. This paper is about third-order delay differential equations and deal with more general equations. We also establish sufficient conditions for the stability of solutions by imposing proper impulse control.

This paper is organized as follows. In Section 2, we establish third-order delay deferential equations. In Section 3, by using Lyapunov function and analysis methods, we prove that the non-impulsive equations can be stabilized by the proper impulse control. In Section 4, two examples are discussed to illustrate the efficiency of the main results.

2. Preliminaries

We consider the following two equations with impulses:

$$\begin{cases} x'''(t)+c(t)x''(t)+b(t)x'(t)+a(t)x(t-\tau)=0, \quad t\ge t_0; \quad t\ne t_k, \quad k=1,2\cdots \\ x(t)=\varphi(t), \quad t_0-\tau\le t\le t_0; \quad x'(t_0)=y_0, \quad x''(t_0)=z_0 \\ x(t_k)=I_k(x(t_k^-)), \quad x'(t_k)=J_k(x'(t_k^-)), \quad x''(t_k)=U_k(x''(t_k^-)) \end{cases} \tag{1}$$

and

$$\begin{cases} x'''(t)+c(t)x''(t)+b(t)x'(t)+\int_{t-\tau}^{t}e(t-u)x(u)du=0, \quad t\ge t_0; \quad t\ne t_k, \quad k=1,2\cdots \\ x(t)=\varphi(t), \quad t_0-\tau\le t\le t_0; \quad x'(t_0)=y_0, \quad x''(t_0)=z_0 \\ x(t_k)=I_k(x(t_k^-)), \quad x'(t_k)=J_k(x'(t_k^-)), \quad x''(t_k)=U_k(x''(t_k^-)) \end{cases} \tag{2}$$

With the following assumptions:

(H_1) $\tau>0,\ x(t):[t_0-\tau,+\infty]\to R$;

(H_2) $x''(t), x'(t)$ denotes the right derivative of $x'(t), x(t)$, and $x''(t)=[x'(t)]'$, $x'(t)=\lim\limits_{h\to 0}\dfrac{x(t+h)-x(t)}{h}$. If $x(t)$ is piecewise continuous, then $x(s^-)$ and $x(s^+)$ denote, respectively, its left and right limits as t tend to s;

(H_3) $\varphi:[t_0-\tau,t_0]\to R$ has at most finite discontinuity points of the first kind and is right continuous at these points;

(H_4) $a(t),b(t),c(t)$ are continuous on $[t_0,+\infty]$, $e(t)$ is continuous on $[0,\tau]$;

(H_5) $0<t_1<t_2<\cdots<t_k<t_{k+1}<\cdots,\quad \lim\limits_{k\to\infty}t_k=+\infty$, with $\tau\le t_{k+1}-t_k\le l$, $t\in N$;

(H_6) consider the impulses at times t_k, $k=1,2\cdots$

$$x(t_k)=I_k(x(t_k^-)),\ x'(t_k)=J_k(x'(t_k^-)),\ x''(t_k)=U_k(x''(t_k^-))$$

Where $I_k,J_k,U_K:R\to R$ are continuous and $I_k(0)=J_k(0)=U_k(0)=0,\ k\in N$;

The following definitions are slightly modified from [1]:

Definition 2.1 A function $x:[t_0-\tau,t_0+\tau)\to R,\ \alpha>0$ is a solution of equation (1)(or(2)), though (t_0,φ,y_0,z_0), if

(i) $x(t), x'(t), x''(t)$ are continuous on $[t_0, t_0+\alpha)\setminus\{t_k, k\in Z\}$;

(ii) $x(t)=\varphi(t),\ t\in[t_0-\tau, t_0],\quad x'(t_0)=y_0,\ x''(t_0)=z_0$, $x(t)$ satisfies the first equality of (1)(or(2) on $[t_0, t_0+\alpha)\setminus\{t_k, k\in Z\}$;

(iii) $x(t), x'(t), x''(t)$ all have two-side limits and right continuous at point t_k, and $x(t_k), x'(t_k), x''(t_k)$ satisfy the third equality of (1)(or(2));

Definition 2.2 The problem of equation (1)(or(2)) is said to be exponentially stabilized by impulses, if there exist $\alpha>0$, a sequence $\{t_k\}_{k\in N}$ satisfying $[H_5]$, and sequences of continuous functions $\{I_k\}, \{J_k\}, \{U_k\}$. such that for all $\varepsilon>0$, there exists $\delta>0$, such that if a solution $x(t;t_0,\varphi,y_0,z_0)$ of (1)(or(2)) fulfills:

$$\sqrt{\|\varphi\|^2+y_0^2+z_0^2}\ \le\delta$$

Then

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)],$$

$t\ge t_0$,

where $\|\varphi(s)\| = \sup\limits_{t_0-\tau\le s\le t_0}|\varphi(s)|$

3 Main Results

First we consider system (1)

Theorem 3.1. If there exist $A\ge 0, B\ge 0, C\ge 0$, such that $|a(t)|\le A,\ |b(t)|\le B,\ |c(t)|\le C,$ and

$$A\tau\le\exp\{-2(1+A+B+C)\tau\},\tag{3}$$

The solution of system(1)can be exponentially stabilized by impulses.

Proof. By (3) there exist $\alpha>0$ and $l\ge\tau$ such that

$$A\tau\le\exp[-2\alpha(l+\tau)]\exp[-2(1+A+B+C)l].\tag{4}$$

Let α, l be as in (4). every sequence $\{t_k\}_{k\in N}$ satisfying (H_5) with $t_k-t_{k-1}\le l,\ \ k\in N$, let

$$I_k(u)=J_k(u)=U_k(u)=d_k u.\quad k=1,2\cdots$$

$$d_k=\sqrt{\frac{P_k-A\tau}{2}}$$

$$p_k=\exp[-2\alpha(t_{k+1}-t_k+\tau)]\exp[-2(1+A+B+C)(t_{k+1}-t_k)]$$

It is easy to verify that $d_k\le 1,$ and

$$d_k^2+A\tau=\frac{p_k+A\tau}{2}\le p_k.$$

For every $\varepsilon>0$, let

$$\delta=\frac{\varepsilon}{\sqrt{1+A\tau}}\exp[-\alpha(t_1-t_0)]\exp[-(1+A+B+C)(t_1-t_0)]$$

We will prove that for each solution $x(t;t_0,\varphi,y_0,z_0)$ of (1), such that

$$\sqrt{\|\varphi\|^2+y_0^2+z_0^2}\le\delta$$

we have

$$\sqrt{x(t)+x'^2(t)+x''^2(t)}\le\varepsilon\exp[-\alpha(t-t_0)]$$

$t\ge t_0$

If $t\in[t_{k-1}, t_k),\ k\in N$, consider the Lyapunov functional

$$V(t)=x^2(t)+x'^2(t)+x''^2(t)+\int_{t-\tau}^{t}|a(s+\tau)|x^2(s)ds$$

and $V(t)$ satisfies:

(i) $V(t)\ge x^2(t)+x'^2(t)+x''^2(t);$

(ii)
$$\begin{aligned}V(t)&\le x^2(t)+x'^2(t)+x''^2(t)+\|x\|_t^2\int_t^{t+\tau}|a(s)|ds\\&\le x^2(t)+x'^2(t)+x''^2(t)+A\tau\|x\|_t^2\\&\le(1+A\tau)\left[\|x\|_t^2+x'^2(t)+x''^2(t)\right]\end{aligned}$$
where $\|x\|_t=\sup\limits_{t-\tau\le s\le t}|x(s)|;$

(iii)
$$\begin{aligned}V'(t)&=2x(t)x'(t)+2x'(t)x''(t)+2x''(t)x'''(t)\\&+|a(t+\tau)|x^2(t)-|a(t)|x^2(t-\tau)\\&=2x(t)x'(t)+2x'(t)x''(t)\\&+2x''(t)[-c(t)x''(t)-b(t)x'(t)-a(t)x(t-\tau)]\\&+|a(t+\tau)|x^2(t)-|a(t)x^2(t-\tau)|\\&\le 2x(t)x'(t)+2x'(t)x''(t)+2c(t)x''^2(t)\\&+2b(t)x'(t)x''(t)+2a(t)\ x(t-\tau)x''(t)\\&+|a(t+\tau)|x^2(t)-|a(t)x^2(t-\tau)|\\&\le[x^2(t)+x'^2(t)]+[x'^2(t)+x''^2(t)]\\&+2Cx''^2(t)+B[x'^2(t)+x''^2(t)]\\&+A[x^2(t-\tau)+x''^2(t)]+Ax^2(t)\\&\le[2(1+A+B+C)][x^2(t)+x'^2(t)+x''^2(t)]\\&\le[2(1+A+B+C)]V(t)\end{aligned}$$

Solving $V'(t)\le[2(1+A+B+C)]V(t)$, we obtain

$$V(t)\le V(t_0)\exp[2(1+A+B+C)(t-t_0)]$$

(1^0) if $t\in[t_0,t_1)$, Integrating the above inequality from t_0 to t_1, we obtain

$$\begin{aligned}&x^2(t)+x'^2(t)+x''^2(t)\le V(t)\\&\le V(t_0)\exp[2(1+A+B+C)(t_1-t_0)]\\&\le(1+A\tau)\left[\|x\|_{t_0}^2+x'^2(t_0)+x''^2(t_0)\right]\\&\exp[2(1+A+B+C)(t_1-t_0)]\end{aligned}$$

$$\le (1+A\tau)\delta^2 \exp\left[2(1+A+B+C)(t_1-t_0)\right]$$
$$\le \varepsilon^2 \exp\left[-2\alpha(t_1-t_0)\right]$$
$$\le \varepsilon^2 \exp\left[-2\alpha(t-t_0)\right]$$

Therefore

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)} \le \varepsilon \exp\left[-\alpha(t-t_0)\right], \quad t\in\left[t_0,t_1\right)$$

Especially

$$\sup_{t_1-\tau\le t\le t_1}\left[x^2(t)+x'^2(t)+x''^2(t)\right]\le \varepsilon^2 \exp\left[-2\alpha(t_1-t_0-\tau)\right]$$

$\left(2^0\right)$ If $t\in\left[t_1,t_2\right)$. by the expression of $V(t)$, we have

$$x^2(t)+x'^2(t)+x''^2(t)\le V(t)$$
$$\le V(t_1^+)\exp\left[2(1+A+B+C)(t-t_1)\right]$$

$$\le V(t_1^+)\exp\left[2(1+A+B+C)(t_2-t_1)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$

$$=\left\{x^2(t_1)+x'^2(t_1)+x''^2(t_1)+\int_{t-\tau}^{t}\left|a(s+\tau)\right|x^2(s)ds\right\} \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$=\left\{\left[I_1(x^2(t_1^-))+J_1(x'^2(t_1^-))+U_1(x''^2(t_1^-))\right] +\int_{t-\tau}^{t}\left|a(s+\tau)\right|x^2(s)\ ds\right\} \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$

$$=\left\{d_1^{\,2}\left[x^2(t_1^-)+x'^2(t_1^-)+x''^2(t_1^-)\right]+\int_{t-\tau}^{t}\left|a(s+\tau)\right|x^2(s)ds\right\} \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$\le d_1^2 \sup_{t_1-\tau\le t\le t_1}\left[x^2(t)+x'^2(t)+x''^2(t)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$

$$+\sup_{t_1-\tau\le t\le t_1} x^2(t)A\tau\exp\left[2(1+A+B+C)(t_2-t_1)\right]$$

$$\le (d_1^2+A\tau)\sup_{t_1-\tau\le t\le t_1}\left[x^2(t)+x'^2(t)+x''^2(t)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$\le (d_1^2+A\tau)\varepsilon^2\exp\left[-2\alpha(t_1-t_0-\tau)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$

By the definitions of d_1 and p_1, we have

$$x^2(t)+x'^2(t)+x''^2(t)\le V(t)$$
$$\le \varepsilon^2(d_1^2+A\tau)\exp\left[-2\alpha(t_1-t_0-\tau)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$=\varepsilon^2\left(\frac{p_1+A\tau}{2}\right)\exp\left[-2\alpha(t_1-t_0-\tau)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$\le \varepsilon^2 p_1\exp\left[-2\alpha(t_1-t_0-\tau)\right] \exp\left[2(1+A+B+C)(t_2-t_1)\right]$$
$$=\varepsilon^2\exp\left[-2\alpha(t_2-t_0)\right]$$
$$\le \varepsilon^2\exp\left[-2\alpha(t-t_0)\right]$$

We obtain for $t\in\left[t_1,t_2\right)$,

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)} \le \varepsilon \exp\left[-\alpha(t-t_0)\right]$$

$\left(3^0\right)$ With analogous arguments,we can verify that for all $k\in N,\ t\in\left[t_{k-1},t_k\right),\ k=1,2\cdots$, we have

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)} \le \varepsilon \exp\left[-\alpha(t-t_0)\right]$$

Hence

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)} \le \varepsilon \exp\left[-\alpha(t-t_0)\right],$$

$t\ge t_0$,

The proof is complete.

Now we prove that the problem (2) can be exponentially stabilized by impulses.

Theorem3.2 If there exist $E\ge 0$, such that $\left|e(t)\right|\le E$, and

$$\frac{1}{2}E\tau^2<\exp\left\{-2(1+A+B+E\tau)\tau\right\}, \tag{5}$$

The solution of system (2) can be exponentially stabilized by impulses.

Proof. By (5), there exist $\alpha>0$ and $l>\tau$, such that

$$\frac{1}{2}E\tau^2<\exp\left[-2\alpha(l+\tau)\right]\exp\left[-2(1+A+B+E\tau)l\right] \tag{6}$$

Let α, l be as in (6). every sequence $\left\{t_k\right\}_{k\in N}$ satisfying (H_5) with $t_k-t_{k-1}\le l,\ k\in N,$ let

$$I_k(u)=J_k(u)=U_k(u)=d_k u.\quad k=1,2\cdots$$

$$d_k=\sqrt{\frac{P_k-\frac{1}{2}E\tau^2}{2}}$$

$$p_k=\exp\left[-2\alpha(t_{k+1}-t_k+\tau)\right] \exp\left[-2(1+A+B+E\tau)(t_{k+1}-t_k)\right]$$

It is easy to verify that $d_k\le 1,$ and

$$d_k^2+\frac{1}{2}E\tau^2=\frac{p_k+\frac{1}{2}E\tau^2}{2}\le p_k.$$

For every $\varepsilon>0$, let

$$\delta=\frac{\varepsilon}{\sqrt{1+\frac{1}{2}E\tau^2}}\exp\left[-\alpha(t_1-t_0)\right] \exp\left[-(1+A+B+E\tau)(t_1-t_0)\right]$$

We will prove that for each solution $x(t;t_0,\varphi,y_0,z_0)$ of (1), such that

$$\sqrt{\|\varphi\|^2 + y_0^2 + z_0^2} \le \delta$$

we have

$$\sqrt{x(t) + x'^2(t) + x''^2(t)} \le \varepsilon \exp[-\alpha(t - t_0)], \quad t \ge t_0,$$

If $t \in [t_{k-1}, t_k)$, $k \in N$, consider the Lyapunov functional

$$V(t) = x^2(t) + x'^2(t) + x''^2(t) + \int_{t-\tau}^{t}\left[\int_u^t |e(u-s+\tau)| x^2(s)ds\right]du$$

and $V(t)$ satisfies

(i) $V(t) \ge x^2(t) + x'^2(t) + x''^2(t)$;

(ii)
$$V(t) \le x^2(t) + x'^2(t) + x''^2(t) + \|x\|_t^2 \int_{t-\tau}^{t}\int_0^{\tau} |e(s)| dsdu$$
$$\le x^2(t) + x'^2(t) + x''^2(t) + \frac{1}{2}E\tau^2 \|x\|_t^2$$
$$\le (1 + \frac{1}{2}E\tau^2)\left[\|x\|_t^2 + x'^2(t) + x''^2(t)\right]$$

where $\|x\|_t = \sup_{t-\tau \le s \le t} |x(s)|$;

(iii)
$$V'(t) = 2x(t)x'(t) + 2x'(t)x''(t) + 2x''(t)x'''(t) + \int_{t-\tau}^{t} |e(u-t+\tau)| x^2(t)du - \int_{t-\tau}^{t} |e(t-s)| x^2(s)ds$$
$$= 2x(t)x'(t) + 2x'(t)x''(t) + 2x''(t)\left[-c(t)x''(t) - b(t)x'(t) - a(t)x(t-\tau)\right] + \int_{t-\tau}^{t} |e(u-t+\tau)| x^2(t)du - \int_{t-\tau}^{t} |e(t-s)| x^2(s)ds$$
$$\le 2x(t)x'(t) + 2x'(t)x''(t) + 2c(t)x''^2(t) + 2b(t)x'(t)x''(t) + 2\int_{t-\tau}^{t} e(t-u)x(u)x''(t)du + \int_{t-\tau}^{t} |e(u-t+\tau)| x^2(t)du - \int_{t-\tau}^{t} |e(t-s)| x^2(s)ds$$
$$\le \left[x^2(t) + x'^2(t)\right] + \left[x'^2(t) + x''^2(t)\right] + 2Cx''^2(t) + B\left[x'^2(t) + x''^2(t)\right] + E\tau\left[x^2(u) + x^2(t)\right] + E\tau x^2(t)$$
$$\le \left[2(1 + A + B + E\tau)\right]\left[x^2(t) + x'^2(t) + x''^2(t)\right]$$
$$\le \left[2(1 + A + B + E\tau)\right]V(t)$$

Solving $V'(t) \le [2(1 + A + B + E\tau)]V(t)$, we obtain

$$V(t) \le V(t_0)\exp[2(1 + A + B + E\tau)(t - t_0)]$$

(1^0) if $t \in [t_0, t_1)$, Integrating the above inequality from t_0 to t_1, we obtain

$$x^2(t) + x'^2(t) + x''^2(t) \le V(t)$$
$$\le V(t_0)\exp[2(1 + A + B + E\tau)(t_1 - t_0)]$$
$$\le (1 + \frac{1}{2}E\tau^2)\left[\|x\|_{t_0}^2 + x'^2(t_0) + x''^2(t_0)\right]\exp[2(1 + A + B + E\tau)(t_1 - t_0)]$$
$$\le (1 + \frac{1}{2}E\tau)\delta^2 \exp[2(1 + A + B + E\tau)(t_1 - t_0)]$$
$$\le \varepsilon^2 \exp[-2\alpha(t_1 - t_0)]$$
$$\le \varepsilon^2 \exp[-2\alpha(t - t_0)]$$

Therefore

$$\sqrt{x^2(t) + x'^2(t) + x''^2(t)} \le \varepsilon \exp[-\alpha(t - t_0)], \quad t \in [t_0, t_1)$$

Especially

$$\sup_{t_1-\tau \le t \le t_1}\left[x^2(t) + x'^2(t) + x''^2(t)\right] \le \varepsilon^2 \exp[-2\alpha(t_1 - t_0 - \tau)]$$

(2^0) If $t \in [t_1, t_2)$. by the expression of $V(t)$, we have

$$x^2(t) + x'^2(t) + x''^2(t) \le V(t)$$
$$\le V(t_1^+)\exp[2(1 + A + B + E\tau)(t - t_1)]$$
$$\le V(t_1^+)\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$= \left\{x^2(t_1^+) + x'^2(t_1^+) + x''^2(t_1^+) + \int_{t-\tau}^{t}\left[\int_u^t |e(u-s+\tau)| x^2(s)ds\right]du\right\}\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$= \left\{x^2(t_1) + x'^2(t_1) + x''^2(t_1) + \int_{t-\tau}^{t}\left[\int_u^t |e(u-s+\tau)| x^2(s)ds\right]du\right\}\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$= \left\{\left[I_1(x^2(t_1^-)) + J_1(x'^2(t_1^-)) + U_1(x''^2(t_1^-))\right] + \int_{t-\tau}^{t}\left[\int_u^t |e(u-s+\tau)| x^2(s)ds\right]du\right\}\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$= \left\{d_1^2\left[x^2(t_1^-) + x'^2(t_1^-) + x''^2(t_1^-)\right] + \int_{t-\tau}^{t}\left[\int_u^t |e(u-s+\tau)| x^2(s)ds\right]du\right\}\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$\le d_1^2 \sup_{t_1-\tau \le t \le t_1}\left[x^2(t) + x'^2(t) + x''^2(t)\right]\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$+ \sup_{t_1-\tau \le t \le t_1} x^2(t)\frac{1}{2}E\tau^2 \exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$\le (d_1^2 + \frac{1}{2}E\tau^2)\sup_{t_1-\tau \le t \le t_1}\left[x^2(t) + x'^2(t) + x''^2(t)\right]\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$
$$\le (d_1^2 + \frac{1}{2}E\tau^2)\varepsilon^2 \exp[-2\alpha(t_1 - t_0 - \tau)]\exp[2(1 + A + B + E\tau)(t_2 - t_1)]$$

By the definitions of d_1 and p_1, we have

$$x^2(t) + x'^2(t) + x''^2(t) \le V(t)$$
$$\le \varepsilon^2(d_1^2 + \frac{1}{2}E\tau^2)\exp[-2\alpha(t_1 - t_0 - \tau)]$$

$$\exp\left[2(1+A+B+E\tau)(t_2-t_1)\right]$$
$$=\varepsilon^2\left(\frac{p_1+\frac{1}{2}E\tau^2}{2}\right)\exp\left[-2\alpha(t_1-t_0-\tau)\right]$$
$$\exp\left[2(1+A+B+E\tau)(t_2-t_1)\right]$$
$$\leq\varepsilon^2 p_1\exp\left[-2\alpha(t_1-t_0-\tau)\right]$$
$$\exp\left[2(1+A+B+E\tau)(t_2-t_1)\right]$$
$$=\varepsilon^2\exp\left[-2\alpha(t_2-t_0)\right]$$
$$\leq\varepsilon^2\exp\left[-2\alpha(t-t_0)\right]$$

We obtain for $t\in\left[t_1,t_2\right)$,

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)}\leq\varepsilon\exp\left[-\alpha(t-t_0)\right]$$

$\left(3^0\right)$ With analogous arguments,we can verify that for all $k\in N,\ t\in\left[t_{k-1},t_k\right),\ k=1,2\cdots$, we have

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)}\leq\varepsilon\exp\left[-\alpha(t-t_0)\right]$$

Hence

$$\sqrt{x^2(t)+x'^2(t)+x''^2(t)}\leq\varepsilon\exp\left[-\alpha(t-t_0)\right],$$

$t\geq t_0$,

The proof is complete.

4 Examples

Example 4.1. Consider the following equation:

$$\begin{cases} x'''(t)+0.33x''(t)-0.025x'(t)-0.5x(t)-x(t-0.01)=0, & t\geq 0 \\ x(t)=\varphi(t), \quad -0.01\leq t\leq 0; \quad x'(0)=y_0, \quad x''(0)=z_0. \end{cases} \tag{7}$$

whose characteristic equation is

$$\lambda^3+0.33\lambda^2-0.025\lambda-0.5-e^{-0.01\lambda}=0$$

By Mathematica sofeware, we find a characteristic root of (7) with the positive real part. Hence the non-impulsive system (7) is unstable.

Consider

$A=1,\quad l=\tau=0.01,\quad \alpha=1/2,\quad B=C=0.5$, and we can verify that

$$A\tau\leq\exp\left[-2\alpha(l+\tau)\right]\exp\left[-2(1+A+B+C)l\right]$$
$$<\exp\left[-2(1+A+B+C)\tau\right].$$

Considering the impulses at t_k, such that $t_k-t_{k-1}\equiv 0.01$ and

$$x(t_k)=I_k(x(t_k^-))=dx(t_k^-),\ \ x'(t_k)=J_k(x'(t_k^-)),$$
$$=dx'(t_k^-),\ \ x''(t_k)=U_k(x''(t_k^-))=dx''(t_k^-)$$

where $d=\sqrt{\dfrac{\exp(-0.08)-0.01}{2}}$, By Theorem 3.1 the unstable system (7) can be exponentially stabilized by impulses.

Example 4.2. Consider the following equation:

$$\begin{cases} x'''(t)-0.75x''(t)-x(t-0.0375)=0, & t\geq 0 \\ x(t)=\varphi(t), \quad -0.0375\leq t\leq 0; \quad x'(0)=y_0, \quad x''(0)=z_0. \end{cases} \tag{8}$$

whose characteristic equation is

$$\lambda^3-0.75\lambda^2-e^{-0.0375\lambda}=0$$

By Mathematica sofeware, we find a characteristic root of (8) with the positive real part. Hence the non-impulsive system (8) is unstable.

Consider

$A=1,\quad l=\tau=0.0375,\quad \alpha=1/2,\quad C=0.75$, and we can verify that

$$A\tau\leq\exp\left[-2\alpha(l+\tau)\right]\exp\left[-2(1+A+B+C)l\right]$$
$$<\exp\left[-2(1+A+B+C)\tau\right].$$

Considering the impulses at t_k, such that $t_k-t_{k-1}\equiv 0.0375$ and

$$x(t_k)=I_k(x(t_k^-))=dx(t_k^-),$$
$$x'(t_k)=J_k(x'(t_k^-))=dx'(t_k^-),$$
$$x''(t_k)=U_k(x''(t_k^-))=dx''(t_k^-),$$

where $d=\sqrt{\dfrac{\exp(-0.28125)-0.0375}{2}}$, By Theorem 3.1 the unstable system (8) can be exponentially stabilized by impulses.

References

[1] A.Weng,J.Sun, Impulsive stabilization of second-order nonlinear delay differential systems,Appl.Math.Comput.214 (2009)95-101

[2] L.P. Gimenes, M. Federson, Existence and impulsive stability for second order retarded differential equations, Appl. Math.Comput.177 (2006)44–62.

[3] Xiang Li, Peixuan Weng, Impulsive stabilization of two kinds of second-order linear delay differential equations, J. Math. Anal. Appl. 291 (2004) 270–281.

[4] A.Weng,J.Sun, Impulsive stabilization of second-order delay differential equations, Nonlinear Anal,:Real Word Appl.8(2007) 1401-1420

[5] L.P. Gimenes, M. Federson, Impulsive stability for systems of second order retarded differential equations, Nonlinear Anal. 67 (2007) 545–553.

[6] Xinzhi Liu, George Ballinger, Existence and continuability of solutions for differential equations with delays and state-dependent impulses, Nonlinear Anal. 51 (2002) 633–647.

[7] L. Berezansky, E. Braverman, Impulsive stabilization of linear delay differential equations, Dynam. Systems,Appl. 5 (1996) 263–276.

[8] W. Feng, Y. Chen, The weak exponential asymptotic stability of impulsive differential system, Appl. Math. J. Chinese Univ. 1 (2002) 1–6.

[9] J. Shen, Z. Luo, X. Liu, Impulsive stabilization of functional differential equations via Liapunov functionals, J. Math. Anal.Appl.240 (1999)1-5

[10] X. Li, Impulsive stabilization of linear differential system, J. South China Normal Univ. Natur. Sci. Ed. 1(2002) 52–56.